The problem is of some historical interest, being due to John Bernoulli. The distinguished American astronomer, G. W. Hill, proposed it in *The Analyst*, Vol. II, January, 1875, p. 31, Ex. 56, with no reference to its history. It was solved by Walter Siverly, and Henry Heaton in the succeeding number of *The Analyst*, by the following method:

By Euler's Theorem, $\cos\theta + i\sin\theta = e^{i\theta}$, where θ may be $(4n+1)(\pi/2)$. Assume $n = 0$. Then $i = e^{(\pi/2)i}$ and $(i)^i = e^{-\frac{1}{2}\pi} = 0.2078795763507$.

Note.—By an oversight, this problem was placed under Calculus.

Solutions were also received from J. B. REYNOLDS, PAUL CAPRON, O. S. ADAMS, and E. B. ESCOTT.

Mr. Escott, using Steinhauser's 20-place tables, gets .20787957635076190854687 while Professor Reynolds, using Hutton's 20-place tables, gets .20787957634917907781.—EDITORS.

MECHANICS.

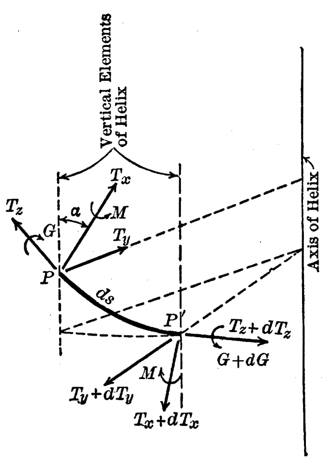
300. Proposed by V. M. SPUNAR, Chicago, Ill.

A helical spring is composed of twenty turns of steel wire 0.258″ in diameter, the diameter of the coil being 3″. If the spring is compressed by a force of 50 lbs., what is the maximum stress in the coil, its axial compression and its resilence?

SOLUTION BY G. PAASWELL, N. Y. City.

This solution follows the method given by Love in his treatise on Elasticity.

At any point, P, of the helix, take a system of rectilinear coördinates chosen as follows: the tangent, as the z axis; the radial element as the y axis; and the normal to the two above, as the x axis. The motion of the curve may be analyzed into a curvature k about the x axis described by the z axis and a twist, t, about the z axis described by the x axis. These are, respectively, if the pitch of the helix is a, $\cos^2 a/r$, $(\sin a \cos a)/r$, where r is the radius of the coil. This curvature, k, causes a bending moment, M, and the twist, t, causes a torsional moment, G.



If s is the total length of the helix; n, the number of turns and h the height of the coil; then $h = s\sin a$ and $s\cos a = 2\pi nr$.

Assume the spring subjected to an elongating force R and consider a small free portion of length ds (see figure). The twist from P to P' is tds and the curvature, kds. From ordinary statics,

$$\Sigma_x = \Sigma_y = \Sigma_z = 0,$$

whence

$$T_x - (T_x + dT_x) + tds(T_y + dT_y) = 0, \qquad (1)$$

$$T_y - (T_y + dT_y) - tds(T_x + dT_x) + kds(T_z + dT_z) = 0, \qquad (2)$$

$$T_z - (T_z + dT_z) - kds(T_y + dT_y) = 0, \qquad (3)$$

$$\Sigma_{\text{mom. }x} = \Sigma_{\text{mom. }y} = \Sigma_{\text{mom. }z} = 0,$$

$$(T_y + dT_y)ds = 0;\ \text{whence } T_y = 0 \text{ and } (T_x + dT_x)ds + Gkds - Mtds = 0.$$

From the above, we get at once $T_x - Mt + Gk = 0$ and $tT_x - kT_z = 0$, or $T_x \tan a = T_z$.

The applied loading deforms the helix into a new one with pitch a' and radius r'. The new curvature and twist is, then, t', k'. Assuming that the original curve was under no stress, the bending and torsional moments are given, respectively, by the difference in the curvatures and twists, multiplied by the flexural and torsional rigidities. Then, $M = EI(k' - k)$ and $G = E_sI_s(t' - t)$, where E is Young's Modulus; E_s is the shearing modulus, I is the moment of inertia of the wire about a diameter; and I_s is the polar moment of inertia of the wire. If Poisson's ratio is taken as 1/4, then $E_s = \frac{4}{5}E$. Representing EI by A, then $M = A(k' - k)$ and $G = \frac{4}{5}A(t' - t)$. (6) can now be written $T_x = A\{t'(k' - k) - \frac{4}{5}k'(t' - t)\}$. Since the resultant stress must equal R, $R = T_x \sec a'$. Likewise, the resultant moment is

$$W = M\cos a' + G\sin a'.$$

There is, finally,

$$R = A\left\{\frac{\sin a'}{r'}\left(\frac{\cos^2 a'}{r'}-\frac{\cos^2 a}{r}\right)-\frac{4}{5}\frac{\cos a'}{r'}\left(\frac{\sin a'\cos a'}{r'}-\frac{\sin a\cos a}{r}\right)\right\}$$

and

$$W = A\left\{\cos a'\left(\frac{\cos^2 a'}{r'}-\frac{\cos^2 a}{r}\right)+\frac{4}{5}\sin a'\left(\frac{\sin a'\cos a'}{r'}-\frac{\sin a\cos a}{r}\right)\right\}.$$

The distortion of the helix will change the total angle φ, turned by the helix into $\varphi + d\varphi$; where φ is $2\pi n$; and the height of the helix from h to $h + dh$. Since $\varphi = (s\cos a)/r$ then $\varphi + d\varphi = (s\cos a')/r$ and $h + dh = s\sin a'$. Taking $d\varphi$ and dh as infinitesimal we have $s\cos a' = s\cos a - \tan a dh$. Since

$$\frac{1}{r} = \frac{\varphi}{\sqrt{s^2-h^2}},$$

then

$$\frac{1}{r^2} = \frac{\varphi + d\varphi}{\sqrt{s^2-(h+dh)^2}}.$$

After proper reductions and substitutions

$$R = \frac{A}{s^3}\left(\frac{4s^2+h^2}{5(s^2-h^2)}\varphi^2 dh - \frac{\varphi h}{5}d\varphi\right) \quad\text{and}\quad W = \frac{A}{S^3}\left(-\frac{\varphi h}{5}dh + \frac{5s^2-h^2}{5}d\varphi\right).$$

There are two possible cases: the terminal is held fast against twist and hence $d\varphi = 0$; and the terminals are free to turn and the resultant moment W is zero. The former is the usual case in practice and will be the only one discussed here. Either condition, with R given, renders the problem a determinate one. Introducing the former condition, then

$$R = \frac{A}{5s^2}\,\frac{4s^2+h^2}{s^2-h^2}\varphi^2 dh \quad\text{and}\quad W = -\frac{A\varphi h}{5S^3}dh.$$

If p is the radius of the wire and S is the unit shear, we have, since the section is a circle,

$$S = pW/I_s.$$

The resilience must equal the work done, whence Res. $= \frac{1}{2}Rdh$.

With the conditions as given in the problem, dh is first to be found, and noting that $s^2 - h^2 = \varphi r^2$, there is

$$dh = \frac{5s^3r^2R}{A(4s^2+h^2)}.$$

Further, since h^2 is usually infinitesimal in comparison with $4s^2$, the formula may be simply written

$$dh = \frac{5sr^2R}{4A}.$$

This is the expression used in the numerical substitution below.

Note, that within reasonable limits of the pitch, s is practically constant so that the shortening of the spring is independent of the pitch.

With the data given in the problem, $r = 1.5''$; $p = 0.129''$; $R = 50$ lbs. and E is assumed as 30,000,000 lbs. so that $A = 6{,}570$, $n = 20$. The horizontal projection of the length of the spring is $2\pi nr$, which is equal to 188.5″, so, that for small pitches, s may be taken about 190. The shortening is then 4.1″. Substituting this in the expression for W and then introducing this value of W for finding S, the maximum stress, *i. e.*, shear, is 855 lbs. per sq. in. The resilience is found to be 102.5 in. lbs.

329. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A smooth circular table is surrounded by a smooth circular rim. Show that the ball, whose coefficient of restitution is e projected along the table from a point in the rim making an angle $\tan^{-1}e^{3/2}$ with the radius through the point, will return to the point of projection after three rebounds.